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ORIGINAL ARTICLE

On a new subclass of bi-univalent functions

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Abstract The purpose of the present paper is to introduce a new subclass of the function class $\sum$ of bi-univalent functions defined in the open unit disc. Furthermore, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions of this class. Relevant connections of the results presented here with various well-known results are briefly indicated.

MATHEMATICS SUBJECT CLASSIFICATION: 30C45

1. Introduction

Let A denote the class of the functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U.

For $n \in N_0$, $0 \leqslant \beta < 1, \lambda \geqslant 0$, we introduce the subclass $Q(n, \lambda, \beta)$ of S of functions of the form (1.1) satisfying the condition

$$\mathrm{Re}\left\{\frac{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)}{z}\right\} > \beta, \quad z \in U, \tag{1.2}$$

where D^n stands for Sălăgean derivative introduced by Sălăgean [1].

For $n = 0$ it reduces to the class $Q_\lambda(\beta)$ studied by Ding et al. [2], (see also [3–6]).

It is well known that every $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f^{-1}(f(\omega)) = \omega, \left(|\omega| < r_0(f),\ r_0(f) \geqslant \frac{1}{4}\right),$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \cdots$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

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Let Σ denote the class of bi-univalent functions in U given by (1.1). For more basic results one may refer Srivastava et al. [7] and references there in.

Brannan and Taha [8] (see also [9]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha(0 \leqslant \alpha < 1)$, respectively (see [10]). Thus, following Branann and Taha [8] (see also [9]), a function $f \in A$ is in the class $S^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order $\alpha(0 \leqslant \alpha < 1)$ if each of the following conditions is satisfied

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 \leqslant \alpha < 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 \leqslant \alpha < 1,\ \omega \in U),$$

where g is the extension of f^{-1} to U. Similarly, a function $f \in A$ is in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order $\alpha(0 \leqslant \alpha < 1)$ if each of the following conditions are satisfied

$$f \in \Sigma \text{ and } \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 \leqslant \alpha < 1,\ z \in U)$$

and

$$\left|\arg\left(1 + \frac{\omega g''(\omega)}{g'(\omega)}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 \leqslant \alpha < 1,\ \omega \in U),$$

where g is the extension of f^{-1} to U. The classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike and bi-convex functions of order α, corresponding (respectively) to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [8,9]).

Recently, several researchers such as ([7,11–13]) obtained the coefficients $|a_2|$, $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ. Motivating with their work, we introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass of the function class Σ employing the techniques used earlier by Srivastava et al. [7] and Frasin and Aouf [11].

In order to prove our main results, we require the following lemma due to [14].

Lemma 1.1. *If $h \in P$ then $|c_k| \leqslant 2$ for each k, where P is the family of all functions h analytic in U for which $Re\{h(z)\} > 0$,*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad \text{for } z \in U.$$

2. Coefficient bounds for the function class $B_\Sigma(n, \alpha, k)$

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $B_\Sigma(n, \alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left\{\frac{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leqslant 1,\ \lambda \geqslant 1,\ z \in U) \tag{2.1}$$

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left\{\frac{(1-\lambda)D^n g(\omega) + \lambda D^{n+1} g(\omega)}{\omega}\right\}\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leqslant 1,\ \lambda \geqslant 1,\ \omega \in U), \tag{2.2}$$

where the function g is given by

$$g(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \tag{2.3}$$

We note that for $n = 0$, $\lambda = 1$ the class $B_\Sigma(n, \alpha, \lambda)$ reduces to the class H^α_Σ introduced and studied by Srivastava et al. [7] and for $n = 0$, the class $B_\Sigma(n, \alpha, \lambda)$ reduces to the class $B_\Sigma(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [11]. We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $B_\Sigma(n, \alpha, \lambda)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(n, \alpha, \lambda)$, $n \in N_0$, $0 < \alpha \leqslant 1$ and $\lambda \geqslant 1$. Then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{4^n(1+\lambda)^2 + \alpha(2.3^n(1+2\lambda) - 4^n(1+\lambda)^2)}} \tag{2.4}$$

and

$$|a_3| \leqslant \frac{2\alpha}{[(1-\lambda)3^n + \lambda 3^{n+1}]} + \frac{4\alpha^2}{[(1-\lambda)2^n + \lambda 2^{n+1}]^2}. \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$\frac{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)}{z} = [p(z)]^\alpha, \tag{2.6}$$

and

$$\frac{(1-\lambda)D^n g(\omega) + \lambda D^{n+1} g(\omega)}{\omega} = [q(\omega)]^\alpha, \tag{2.7}$$

where $p(z)$ and $q(\omega)$ in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.8}$$

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + q_3\omega^3 + \cdots. \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we obtain

$$[(1-\lambda)2^n + \lambda 2^{n+1}]a_2 = \alpha p_1 \tag{2.10}$$

$$[(1-\lambda)3^n + \lambda 3^{n+1}]a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.11}$$

$$-[(1-\lambda)2^n + \lambda 2^{n+1}]a_2 = \alpha q_1 \tag{2.12}$$

$$[(1-\lambda)3^n + \lambda 3^{n+1}](2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we obtain

$$p_1 = -q_1 \tag{2.14}$$

and

$$2[(1-\lambda)2^n + \lambda 2^{n+1}]^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.15}$$

Now from 2.11, 2.13 and 2.15, we obtain

$$2[(1-\lambda)3^n+\lambda 3^{n+1}]a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$
$$=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\frac{2[(1-\lambda)2^n+\lambda 2^{n+1}]^2}{\alpha^2}a_2^2.$$

Therefore we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{4^n(1+\lambda)^2+\alpha[2.3^n(1+2\lambda)-4^n(1+\lambda)^2]}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\leqslant\frac{2\alpha}{\sqrt{4^n(1+\lambda)^2+\alpha[2.3^n(1+2\lambda)-4^n(1+\lambda)^2]}}$$

Next, in order to find the bound on $|a_3|$ by subtracting (2.13) from (2.11), we obtain

$$2[(1-\lambda)3^n+\lambda 3^{n+1}][a_3-a_2^2]=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

$$2[(1-\lambda)3^n+\lambda 3^{n+1}]a_3=\alpha(p_2-q_2)+\frac{2[(1-\lambda)3^n+\lambda 3^{n+1}]\alpha^2(p_1^2+q_1^2)}{2[(1-\lambda)2^n+\lambda 2^{n+1}]^2}$$

$$a_3=\frac{\alpha(p_2-q_2)}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}+\frac{\alpha^2(p_1^2+q_1^2)}{2[(1-\lambda)2^n+\lambda 2^{n+1}]^2}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$|a_3|\leqslant\frac{2\alpha}{[(1-\lambda)3^n+\lambda 3^{n+1}]}+\frac{4\alpha^2}{[(1-\lambda)2^n+\lambda 2^{n+1}]^2}.$$

This completes the proof of Theorem 2.1. □

3. Coefficient bounds for the function class $H_\Sigma(n, \beta, k)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma(n,\beta,\lambda)$ if the following conditions are satisfied:

$$f\in\Sigma \quad\text{and}\quad Re\left\{\frac{(1-\lambda)D^nf(z)+\lambda D^{n+1}f(z)}{z}\right\}>\beta,$$
$$(0<\beta\leqslant 1,\ \lambda\geqslant 1,\ n\in N_0,\ z\in U) \tag{3.1}$$

and

$$Re\left\{\frac{(1-\lambda)D^ng(\omega)+\lambda D^{n+1}g(\omega)}{\omega}\right\}>\beta,$$
$$(0<\beta\leqslant 1,\ \lambda\geqslant 1,\ n\in N_0,\ z\in U), \tag{3.2}$$

where the function g is defined by (2.3).

We note that for $n = 0$ and $n = 0$, $\lambda = 1$, the class $H_\Sigma(n,\beta,\lambda)$ reduce to the classes $H_\Sigma(\beta,\lambda)$ and $H_\Sigma(\lambda)$ studied by Frasin and Aouf [11] and Srivastava et al. [7], respectively.

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma(n,\beta,\lambda)$, $n\in N_0$, $0\leqslant\beta<1$ and $\lambda\geqslant 1$. Then*

$$|a_2|\leqslant\sqrt{\frac{2(1-\beta)}{[(1-\lambda)3^n+\lambda 3^{n+1}]}} \tag{3.3}$$

and

$$|a_3|\leqslant\frac{4(1-\beta)^2}{[(1-\lambda)2^n+\lambda 2^{n+1}]^2}+\frac{2(1-\beta)}{[(1-\lambda)3^n+\lambda 3^{n+1}]}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exists $p(z)\in P$ and $q(z)\in P$ such that

$$\frac{(1-\lambda)D^nf(z)+\lambda D^{n+1}f(z)}{z}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$\frac{(1-\lambda)D^ng(\omega)+\lambda D^{n+1}g(\omega)}{\omega}=\beta+(1-\beta)q(\omega), \tag{3.6}$$

where $p(z)$ and $q(\omega)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$[(1-\lambda)2^n+\lambda 2^{n+1}]a_2=(1-\beta)p_1, \tag{3.7}$$
$$[(1-\lambda)3^n+\lambda 3^{n+1}]a_3=(1-\beta)p_2, \tag{3.8}$$
$$-[(1-\lambda)2^n+\lambda 2^{n+1}]a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$[(1-\lambda)3^n+\lambda 3^{n+1}](2a_2^2-a_3)=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we have

$$p_1=-q_1 \tag{3.11}$$

and

$$2[(1-\lambda)2^n+\lambda 2^{n+1}]^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$2[(1-\lambda)3^n+\lambda 3^{n+1}]a_2^2=(1-\beta)(p_2+q_2). \tag{3.13}$$

or

$$a_2^2=\frac{(1-\beta)(p_2+q_2)}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}$$
$$|a_2^2|\leqslant\frac{2(1-\beta)}{[(1-\lambda)3^n+\lambda 3^{n+1}]},$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$ by subtracting (3.10) from (3.8), we obtain

$$2[(1-\lambda)3^n+\lambda 3^{n+1}][a_3-a_2^2]=(1-\beta)(p_2-q_2)$$

or, equivalently

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}.$$

Upon substituting the value of a_2^2 from (3.12), we have

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{2[(1-\lambda)2^n+\lambda 2^{n+1}]}+\frac{(1-\beta)(p_2-q_2)}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2[(1-\lambda)2^n+\lambda 2^{n+1}]^2} + \frac{(1-\beta)(p_2-q_2)}{2[(1-\lambda)3^n+\lambda 3^{n+1}]}.$$

$$|a_3| \leqslant \frac{4(1-\beta)^2}{[(1-\lambda)2^n+\lambda 2^{n+1}]^2} + \frac{2(1-\beta)}{[(1-\lambda)3^n+\lambda 3^{n+1}]},$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Remark 3.1. If we put $n = 0$ in Theorems 2.1 and 3.1, we obtain the corresponding results due to Frasin and Aouf [11].

Remark 3.2. If we put $n = 0$, $\lambda = 1$ in Theorems 2.1 and 3.1, we obtain the corresponding results due to Srivastava et al. [7].

Remark 3.3. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|$, $n \geqslant 4$.

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